# Group Analysis of Differential Equations and Generalized Functions

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**Abstract.** We present an extension of the methods of classical Lie group analysis of differential equations to equations involving generalized functions (in particular: distributions). A suitable framework for such a generalization is provided by Colombeau's theory of algebras of generalized functions. We show that under some mild conditions on the differential equations, symmetries of classical solutions remain symmetries for generalized solutions. Moreover, we introduce a generalization of the infinitesimal methods of group analysis that allows to compute symmetries of linear and nonlinear differential equations containing generalized function terms. Thereby, the group generators and group actions may be given by generalized functions themselves.

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# 1 Introduction

Symmetry properties of distributions and group invariant distributional solutions (in particular: fundamental solutions) to particular types of linear differential operators have been studied by Methée ([22]), Tengstrand ([36]), Szmydt and Ziemian ([33, 34, 35], [38]). A systematic investigation of the transfer of classical group analysis of differential equations into a distributional setting is due to Berest and Ibragimov ([2, 3, 4, 5], [18]), again with a view to determining fundamental solutions of certain linear partial differential equations. A survey of the lastnamed studies including a comprehensive bibliography can be found in the third volume of [19]. As these approaches use methods from classical distribution theory, their range is confined to linear equations and linear transformations of the dependent variables.

Algebras of generalized functions offer the possibility of going beyond these limitations towards a generalization of group analysis to genuinely nonlinear problems involving singular terms, like distributions or discontinuous nonlinearities. In the present paper we develop a theory of group analysis of differential equations in algebras of generalized functions that allows a satisfactory treatment of such problems. This line of research has been initiated in [28] and has been taken up in [21].

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Applications to different types of algebras of generalized functions can be found in [30] and [31].

The plan of the paper is as follows: Section 2 provides a short introduction to the theory of algebras of generalized functions in the sense of J.F. Colombeau. In section 3 we consider systems of partial differential equations together with a classical symmetry group $G$ that transforms smooth solutions into smooth solutions. Assuming polynomial bounds on the action of $G$, we can extend it to generalized functions belonging to Colombeau algebras and ask whether $G$ remains a symmetry group for generalized solutions. In section 3.1 we develop methods based on a factorization property of the transformed system of equations. Essentially, polynomial bounds on the factors suffice to give a positive answer. In the scalar case we show this to be automatically satisfied whenever the equation contains at least one of the derivatives of the solution as an isolated term. While the conditions of section 3.1 concern some mild assumptions on the algebraic structure of the equations, section 3.2 develops a topological criterion, applicable to systems of linear equations: the existence of a $\mathcal{C}^\infty$-continuous homogeneous right inverse guarantees a positive answer as well. Along the way we give examples of nonlinear symmetry transformations of shock and delta wave solutions to linear and nonlinear systems. The purpose of section 4 is to develop the general theory, allowing the equations and the group action (hence also its generators) to be given by generalized functions. Using the characterization of Colombeau generalized functions by their generalized pointvalues established in [27] as well as results on Colombeau solutions to ODEs, we show that the classical procedure for computing symmetries can be literally transferred to the generalized function situation. The defining equations are derived as usual, but their solutions are sought in generalized functions. This enlarges the reservoir of possible symmetries of classical equations and allows the study of symmetries of equations with singular terms. An example is provided by a conservation law with discontinuous flux function.

The remainder of the introduction is devoted to fixing notations and recalling some basic definitions from group analysis of differential equations. We basically follow the notations and terminology of [29]. Thus for the action of a Lie group $G$ on some manifold $M$, assumed to be an open subset of some space $\mathcal{X} \times \mathcal{U}$ of independent and dependent variables (with $\dim(\mathcal{X}) = p$ and $\dim(\mathcal{U}) = q$) we write $g \cdot (x, u) = (\Xi_g(x, u), \Phi_g(x, u))$. Transformation groups are always supposed to act regularly on $M$. If $\Xi_g$ does not depend on $u$, the group action is called projectable. Elements of the Lie algebra $\mathfrak{g}$ of $G$ as well as the corresponding vector fields on $M$ will typically be denoted by $\mathbf{v}$ and the one-parameter subgroup generated by $\mathbf{v}$ by $\eta \to \exp(\eta\mathbf{v})$. $M^{(n)}$ denotes the $n$-jet space of $M$; the $n$-th prolongation of a group action $g$ or vector field $\mathbf{v}$ is written as $\mathrm{pr}^{(n)} g$ or $\mathrm{pr}^{(n)}\mathbf{v}$, respectively. Any system $S$ of $n$-th order differential equations in $p$ dependent and $q$ independent variables can be written in the form

$$\Delta_\nu(x, u^{(n)}) = 0, \quad 1 \leq \nu \leq l.$$

where the map

$$\Delta : \mathcal{X} \times \mathcal{U}^{(n)} \to \mathbb{R}^l$$
$$(x, u^{(n)}) \to (\Delta_1(x, u^{(n)}), \ldots, \Delta_l(x, u^{(n)}))$$

will be supposed to be smooth. Hence $S$ is identified with the subvariety

$$S_\Delta = \{(x, u^{(n)}) : \Delta(x, u^{(n)}) = 0\}$$

of $\mathcal{X} \times \mathcal{U}^{(n)}$. For any $f : \Omega \subseteq \mathcal{X} \to \mathcal{U}$, $\Gamma_f$ is the graph of $f$ and $\Gamma_f^{(n)} := \{(x, \mathrm{pr}^{(n)} f(x)) : x \in \Omega\}$ is the graph of the $n$-jet of $f$.

# 2 Colombeau algebras

Already at a very early stage of development of the theory of distributions it became clear that it is impossible to embed the space $\mathcal{D}'(\Omega)$ of distributions over some open subset $\Omega$ of $\mathbb{R}^n$ into an associative commutative algebra $(\mathcal{A}(\Omega), +, \circ)$ satisfying

(i) $\mathcal{D}'(\Omega)$ is linearly embedded into $\mathcal{A}(\Omega)$ and $f(x) \equiv 1$ is the unity in $\mathcal{A}(\Omega)$.

(ii) There exist derivation operators $\partial_i : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ $(i = 1, \ldots, n)$ that are linear and satisfy the Leibniz rule.

(iii) $\partial_i|_{\mathcal{D}'(\Omega)}$ is the usual partial derivative $(i = 1, \ldots, n)$.

(iv) $\circ|_{\mathcal{C}(\Omega) \times \mathcal{C}(\Omega)}$ coincides with the pointwise product of functions.

(Schwartz's impossibility result, [32]). Furthermore, replacing $\mathcal{C}(\Omega)$ by $\mathcal{C}^{(k)}(\Omega)$ does not alter this result. On the other hand, many problems involving differentiation and nonlinearities in the presence of singular objects require a method of coping with this situation in a consistent manner (cf. e.g. [24], [26], [37]). By the above, the best possible result would consist in constructing an algebra $\mathcal{A}(\Omega)$ satisfying (i)–(iii) and

(iv') $\circ|_{\mathcal{C}^\infty(\Omega) \times \mathcal{C}^\infty(\Omega)}$ coincides with the pointwise product of functions.

The actual construction of algebras enjoying these optimal properties is due to J.F. Colombeau ([8], [9], see also [1], [24]). The basic idea underlying his theory (in its simplest – the so-called 'special' – form) is that of embedding the space of distributions into a factor algebra of $\mathcal{C}^\infty(\Omega)^I$ $(I = (0, 1])$ via regularization by convolution with a fixed 'mollifier' $\rho \in \mathcal{S}(\mathbb{R}^n)$ with $\int \rho(x)\, dx = 1$. In order to motivate the definition below let $\rho_\varepsilon(x) := \varepsilon^{-n} \rho(\frac{x}{\varepsilon})$ and let $u \in \mathcal{E}'(\mathbb{R}^n)$ (the space of compactly supported distributions on $\mathbb{R}^n$). The sequence $(u * \rho_\varepsilon)_{\varepsilon \in I}$ converges to $u$ in $\mathcal{D}'(\mathbb{R}^n)$. Taking this sequence as a representative of $u$ we obtain an embedding of $\mathcal{D}'(\mathbb{R}^n)$ into the algebra $\mathcal{C}^\infty(\mathbb{R})^I$. However, embedding $\mathcal{C}^\infty(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$ into

this algebra via convolution as above will not yield a subalgebra since of course $(f*\rho_\varepsilon)(g*\rho_\varepsilon)\neq(fg)*\rho_\varepsilon$ in general. The idea, therefore, is to factor out an ideal $\mathcal{N}(\mathbb{R}^n)$ such that this difference vanishes in the resulting quotient. In order to construct $\mathcal{N}(\mathbb{R}^n)$ it is obviously sufficient to find an ideal containing all differences $(f*\rho_\varepsilon)_{\varepsilon\in I}-(f)_{\varepsilon\in I}$. Taylor expansion of $f*\rho_\varepsilon-f$ shows that this term will vanish faster than any power of $\varepsilon$, (uniformly on compact sets, in all derivatives) provided we additionally suppose that $\int\rho(x)x^\alpha\,dx=0$ for all $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\geq 1$. The set of all such sequences is not an ideal in $\mathcal{C}^\infty(\mathbb{R}^n)^I$, so we shall replace $\mathcal{C}^\infty(\mathbb{R}^n)^I$ by the set of *moderate* sequences $\mathcal{E}_M(\mathbb{R}^n)$ whose every derivative is bounded uniformly on compact sets by some inverse power of $\varepsilon$.
Thus we define the Colombeau algebra $\mathcal{G}(\Omega)$ as the quotient algebra $\mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$, where

$$\begin{aligned}
\mathcal{E}_M(\Omega) &:= \{(u_\varepsilon)_{\varepsilon\in I}\in\mathcal{C}^\infty(\Omega)^I : \forall K\subset\subset\Omega, \forall\alpha\in\mathbb{N}_o^n\ \exists p\in\mathbb{N} \text{ with}\\
&\qquad \sup_{x\in K}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-p}) \text{ as } \varepsilon\to 0\}\\
\mathcal{N}(\Omega) &:= \{(u_\varepsilon)_{\varepsilon\in I}\in\mathcal{C}^\infty(\Omega)^I : \forall K\subset\subset\Omega, \forall\alpha\in\mathbb{N}_o^n\ \forall q\in\mathbb{N}\\
&\qquad \sup_{x\in K}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{q}) \text{ as } \varepsilon\to 0\}.
\end{aligned}$$

Equivalence classes of sequences $(u_\varepsilon)_{\varepsilon\in I}$ in $\mathcal{G}(\Omega)$ will be denoted by $\mathrm{cl}[(u_\varepsilon)_{\varepsilon\in I}]$. $\mathcal{G}(\Omega)$ is a differential algebra containing $\mathcal{E}'(\Omega)$ as a linear subspace via the embedding $\iota: u\to\mathrm{cl}[(u*\rho_\varepsilon)_{\varepsilon\in I}]$ depending on a mollifier $\rho\in\mathcal{S}(\mathbb{R}^n)$ as above. $\iota$ commutes with partial derivatives and coincides with $u\to\mathrm{cl}[(u)_{\varepsilon\in I}]$ on $\mathcal{D}(\Omega)$, thus rendering it a faithful subalgebra of $\mathcal{G}(\Omega)$. The functor $\Omega\to\mathcal{G}(\Omega)$ is a fine sheaf of differential algebras on $\mathbb{R}^n$ and there is a unique sheaf morphism $\hat{\iota}$ extending the above embedding to $\mathcal{C}^\infty(\,.\,)\hookrightarrow\mathcal{D}'(\,.\,)\hookrightarrow\mathcal{G}(\,.\,)$. $\hat{\iota}$ commutes with partial derivatives, and its restriction to $\mathcal{C}^\infty$ is a sheaf morphism of algebras.
We shall also consider the algebra $\mathcal{G}_\tau(\Omega)=\mathcal{E}_\tau(\Omega)/\mathcal{N}_\tau(\Omega)$ of tempered generalized functions, where

$$\begin{aligned}
&\mathcal{O}_M(\Omega)=\{f\in\mathcal{C}^\infty(\Omega):\forall\alpha\in\mathbb{N}_o^n\ \exists p>0\ \sup_{x\in\Omega}(1+|x|)^{-p}|\partial^\alpha f(x)|<\infty\}\\
&\mathcal{E}_\tau(\Omega)=\{(u_\varepsilon)_{\varepsilon\in I}\in(\mathcal{O}_M(\Omega))^I:\forall\alpha\in\mathbb{N}_o^n\ \exists p>0\\
&\qquad\qquad \sup_{x\in\Omega}(1+|x|)^{-p}|\partial^\alpha u_\varepsilon(x)|=O(\varepsilon^{-p})\ (\varepsilon\to 0)\}\\
&\mathcal{N}_\tau(\Omega)=\{(u_\varepsilon)_{\varepsilon\in I}\in(\mathcal{O}_M(\Omega))^I:\forall\alpha\in\mathbb{N}_o^n\ \exists p>0\ \forall\ q>0\\
&\qquad\qquad \sup_{x\in\Omega}(1+|x|)^{-p}|\partial^\alpha u_\varepsilon(x)|=O(\varepsilon^{q})\ (\varepsilon\to 0)\}
\end{aligned}$$

The map $\iota$ defined above is a linear embedding of $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{G}_\tau(\mathbb{R}^n)$ commuting with partial derivatives and making

$$\mathcal{O}_C(\mathbb{R}^n)=\{f\in\mathcal{C}^\infty(\mathbb{R}^n):\exists p>0\ \forall\alpha\in\mathbb{N}_o^n\ \sup_{x\in\mathbb{R}^n}(1+|x|)^{-p}|\partial^\alpha f(x)|<\infty\}$$

a faithful subalgebra. Elements of $\mathcal{O}_M(\Omega)$ are called *slowly increasing*. Componentwise insertion of elements of $\mathcal{G}$ into slowly increasing functions yields well defined elements of $\mathcal{G}$. Thus, in $\mathcal{G}$ not only polynomial combinations of distributions (e.g. $\delta^2$) make sense but also expressions like $\sin(\delta)$ have a well-defined meaning. The importance of $\mathcal{G}_\tau(\Omega)$ for our purposes stems from the fact that elements of this algebra can even be composed with each other (again by componentwise insertion, cf. [16], [20]), a necessary prerequisite for generalizing symmetry methods, see section 4. Especially in the theory of ODEs in the generalized function context it is often useful to consider the algebra $\widetilde{\mathcal{G}}_\tau(\Omega \times \Omega')$ whose elements satisfy $\mathcal{G}$-bounds in the $\Omega$-variables and $\mathcal{G}_\tau$-bounds in the $\Omega'$-variables (cf. [16] or [20]). Elements of Colombeau algebras are usually denoted by capital letters with the understanding that $(u_\varepsilon)_{\varepsilon \in I}$ denotes an arbitrary representative of $U \in \mathcal{G}$.
Nonlinear operations with distributions in $\mathcal{G}(\Omega)$ depend not only on the distributions themselves but also on the regularization procedure used in the embedding process. Thus the difference of two representatives $(u_\varepsilon)_{\varepsilon \in I}$, $(v_\varepsilon)_{\varepsilon \in I}$ of generalized functions $U$ resp. $V$ may have $\mathcal{D}'$-limit 0 as $\varepsilon \to 0$ without $U$ and $V$ being equal in $\mathcal{G}(\Omega)$. Nevertheless $U$ and $V$ are to be considered 'equal in the sense of distributions' or *associated* with each other ($U \approx V$). Moreover, $U$ is called associated with some distribution $w$ if $u_\varepsilon \to w$ in $\mathcal{D}'$. If such a $w$ exists (which need not be the case, cf. $\delta^2$), it is to be seen as the distributional 'shadow' of $U$. For example, all powers of the Heaviside function are associated with each other without being equal in the algebra itself. Also, $x\delta = 0$ in $\mathcal{D}'(\mathbb{R})$, so $x\delta \approx 0$ in $\mathcal{G}(\mathbb{R})$, but $x\delta \neq 0$ in $\mathcal{G}(\mathbb{R})$. These examples illustrate a general principle: assigning nonlinear properties to elements of the vector space $\mathcal{D}'(\Omega)$ amounts to introducing additional information which is reflected in a more rigid concept of equality within $\mathcal{G}(\Omega)$ compared to that in $\mathcal{D}'(\Omega)$. This strict concept of equality allows for much more refined ways of infinitesimal modelling. On the $\mathcal{D}'$-level (the level of association) this additional information is lost in the limit-process $\varepsilon \to 0$.
Generalized numbers (i.e. the ring of constants in case $\Omega$ is connected) in any of the above algebras will be denoted by $\mathcal{R}$. Componentwise insertion of points into representatives of generalized functions yields well defined elements of $\mathcal{R}$.
We note that there exist variants of Colombeau algebras that allow a canonical embedding of distributions (indepenent of a fixed mollifier as above). The basic idea for constructing these algebras is to replace the index set $I$ by the space of *all* possible mollifiers. Our choice of the special variants of Colombeau algebras is aimed at notational simplicity. However, all results presented in the sequel carry over to the respective full variants of the algebras. Moreover, recently there have been introduced global versions of Colombeau algebras, defined intrinsically on manifolds and displaying the analogues of (i)–(iv) (with $\partial_i$ replaced by Lie-derivatives with respect to smooth vector fields), see [14]. For applications of the theory to nonlinear PDEs see [24] and the literature cited therein, for applications to mathematical physics and numerics, cf. [6], [10] and [37].

# 3 Transfer of Classical Symmetry Groups

## 3.1 Factorization Properties

The first question to be answered in trying to extend the applicability of classical group analysis to generalized solutions concerns permanence properties of classical symmetries: Let $G$ be the symmetry group of some system $S$ of PDEs and consider $S$ within the framework of $\mathcal{G}(\Omega)$. Under which conditions do elements of $G$ also transform generalized solutions into other generalized solutions? It is the aim of this and the following section to answer this question. To begin with, let us fix some terminology:

**3.1 Definition** *Let $G$ be a projectable local group of transformations acting on some open set $\mathcal{M} \subseteq \mathcal{X} \times \mathcal{U}$ according to $g \cdot (x, u) = (\Xi_g(x), \Phi_g(x, u))$. $g$ is called slowly increasing if the map $u \to \Phi_g(x, u)$ is slowly increasing, uniformly for $x$ in compact sets. $g$ is strictly slowly increasing if $\Phi_g \in \mathcal{O}_M(\mathcal{M})$. If $\Omega \subseteq \mathcal{X}$, $U \in \mathcal{G}(\Omega)$ and $g$ is (strictly) slowly increasing, the action of $g$ on $U$ is defined as the element*

$$gU := \mathrm{cl}[((\Phi_g \circ (id \times u_\varepsilon)) \circ \Xi_g^{-1})_{\varepsilon \in I}] \tag{1}$$

*of $\mathcal{G}(\Xi_g(\Omega))$.*

If $U$ is a smooth function, (1) reproduces the classical notion of group action on functions. Henceforth we make the tacit assumption that the differential equations under consideration are of a form that allows for an insertion of elements of Colombeau generalized functions (i.e. the function $\Delta$ representing the equations on the prolongation space is slowly increasing). Also, slowly increasing group actions are always understood to be projectable. Analogous to the classical setting we give the following

**3.2 Definition** *Let $S$ be some system of differential equations with $p$ variables and $q$ unknown functions. A solution of $S$ in $\mathcal{G}$ is an element $U \in (\mathcal{G}(\Omega))^q$, with $\Omega \subseteq \mathcal{X}$ open, which solves the system with equality in $(\mathcal{G}(\Omega))^l$. A symmetry group of $S$ in $\mathcal{G}$ is a local transformation group acting on $\mathcal{X} \times \mathcal{U}$ such that if $U$ is a solution of the system in $\mathcal{G}$, $g \in G$ and $g \cdot U$ is defined, then also $g \cdot U$ is a solution of $S$ in $\mathcal{G}$.*

Let us take a look at the transition problem from classical to generalized symmetry groups on the level of representatives. Thus, let $G$ be a slowly increasing symmetry group of some differential equation

$$\Delta(x, u^{(n)}) = 0. \tag{2}$$

This means that if $f$ is a classical solution, i.e. if $\Delta(x, \mathrm{pr}^{(n)} f(x)) = 0$ for all $x$ then also $\Delta(x, \mathrm{pr}^{(n)}(g \cdot f)(x)) = 0$. Now let $U \in \mathcal{G}(\Omega)$ be a generalized solution to (2).

Then for any representative $(u_\varepsilon)_{\varepsilon\in I}$ of $U$ there exists some $(n_\varepsilon)_{\varepsilon\in I} \in \mathcal{N}(\Omega)$ such that for all $x$ and all $\varepsilon$ we have

$$\Delta(x, \mathrm{pr}^{(n)} u_\varepsilon(x)) = n_\varepsilon(x). \tag{3}$$

In particular, the differential equation (2) need not be satisfied for even one single value of $\varepsilon$. This basic observation displays quite fundamental obstacles to a direct utilization of the classical symmetry group properties of $G$ in order to obtain statements on the status of $G$ in the Colombeau-setting. Therefore we have to derive properties of symmetry groups that are better suited to allow a transfer to differential algebras. The starting point for our considerations is a slight modification of a well known factorization property of smooth maps (cf. [29] , Proposition 2.10):

**3.3 Proposition** *Let $F$ be a smooth mapping from some manifold $M$ to $\mathbb{R}^k$ ($k \le n = dim(M)$), let $f : (-\eta_o, \eta_o) \times M \to \mathbb{R}$ be smooth and suppose that $f(\eta, .)$ vanishes on the zero set $\mathcal{S}_F$ of $F$, identically in $\eta$. If $F$ is of maximal rank ($= k$) on $\mathcal{S}_F$ then there exist smooth functions $Q_1, \ldots, Q_k : (-\eta_o, \eta_o) \times M \to \mathbb{R}$ such that*

$$f(\eta, m) = Q_1(\eta, m)F_1(m) + \ldots + Q_k(\eta, m)F_k(m)$$

*for all $(\eta, m) \in (-\eta_o, \eta_o) \times M$.* □

We are mainly interested in the following application of Proposition 3.3:

**3.4 Theorem** *Let*

$$\Delta_\nu(x, u^{(n)}) = 0, \quad 1 \le \nu \le l \tag{4}$$

*be a nondegenerate system of PDEs. Let $G = \{g_\eta : \eta \in (-\eta_o, \eta_o)\}$ be a one parameter symmetry group of (4) and set $g_\eta \cdot (x, u) = (\Xi_\eta(x, u), \Phi_\eta(x, u))$. Then there exist $\mathcal{C}^\infty$-functions $Q_{\mu\nu} : (-\eta_o, \eta_o) \times \mathcal{V} \to \mathbb{R}$ ($1 \le \mu, \nu \le l$, $\mathcal{V}$ an open subset of $\mathcal{M}^{(n)}$) such that if $u : \Omega \subseteq \mathbb{R}^p \to \mathbb{R}^q$ is smooth and $g_\eta u$ exists we have*

$$\begin{aligned} &\Delta_\nu(\Xi_\eta(x, u(x)), \mathrm{pr}^{(n)}(g_\eta u)(\Xi_\eta(x, u(x)))) = \\ &= \sum_{\mu=1}^{l} Q_{\mu\nu}(\eta, x, \mathrm{pr}^{(n)} u(x)) \Delta_\mu(x, \mathrm{pr}^{(n)} u(x)) \end{aligned} \tag{5}$$

*on the domain of $g_\eta u$ for $1 \le \nu \le l$.*

**Proof.** Denote by $z$ the coordinates on $\mathcal{M}^{(n)}$. That $g_\eta$ is an element of the symmetry group of the system is equivalent with

$$\Delta(z) = 0 \Rightarrow \Delta_\nu(\mathrm{pr}^{(n)} g_\eta(z)) = 0 \quad (1 \le \nu \le l)$$

for all $\eta$ and $z$ such that this is defined. $\Delta$ is of maximal rank because (4) is nondegenerate. Hence, by Proposition 3.3 there exist $\mathcal{C}^\infty$-functions $Q_{\mu\nu} : (-\eta_o, \eta_o) \times \mathcal{V} \to \mathbb{R}$ ($1 \le \mu \le l$, $\mathcal{V}$ an open subset of $\mathcal{M}^{(n)}$) such that

$$\Delta_\nu(\mathrm{pr}^{(n)} g_\eta(z)) = \sum_{\mu=1}^{l} Q_{\mu\nu}(\eta, z) \Delta_\mu(z). \tag{6}$$

Now for a smooth function $u : \Omega \subseteq \mathbb{R}^p \to \mathbb{R}^q$ as in our assumption and $x \in \Omega$ we set

$$z_u(x) := (x, \mathrm{pr}^{(n)} u(x)) \in \mathcal{M}^{(n)}. \tag{7}$$

Then by definition $\mathrm{pr}^{(n)} g_\eta(z_u(x)) = (\Xi_\eta(x, u(x)), \mathrm{pr}^{(n)}(g_\eta u)(\Xi_\eta(x, u(x))))$, so the result follows. □

For a single PDE $\Delta(x, \mathrm{pr}^{(n)} u) = 0$, equation (5) takes the simpler form

$$\Delta(\Xi_\eta(x, u(x)), \mathrm{pr}^{(n)}(g_\eta u)(\Xi_\eta(x, u(x)))) = Q(\eta, x, \mathrm{pr}^{(n)} u(x)) \Delta(x, \mathrm{pr}^{(n)} u(x)). \tag{8}$$

Theorem 3.4 will be one of our main tools in transferring classical symmetry groups of (systems of) PDEs into the setting of algebras of generalized functions.

**3.5 Proposition** *Let $\eta \to g_\eta$ be a slowly increasing one parameter symmetry group of (4). If $P_{\mu\nu} := (Q_{\mu\nu}(\eta, \Xi_{-\eta}(\,.\,), \mathrm{pr}^{(n)} u_\varepsilon(\Xi_{-\eta}(\,.\,))))_{\varepsilon \in I}$ belongs to $\mathcal{E}_M(\Omega)$ for $1 \le \mu, \nu \le l$ and every $(u_\varepsilon)_{\varepsilon \in I} \in \mathcal{E}_M(\Omega)$, then $\eta \to g_\eta$ is a symmetry group of (4) in $\mathcal{G}$ as well. This condition is satisfied if*

$$(x, u^{(n)}) \to Q_{\mu\nu}(\eta, x, u^{(n)})$$

*is slowly increasing in the $u^{(n)}$-variables, uniformly in $x$ on compact sets for $1 \le \mu, \nu \le l$ and every $\eta$.*

**Proof.** It suffices to observe that (5) gives

$$\begin{aligned} &\Delta_\nu(x, \mathrm{pr}^{(n)}(g_\eta u)(x)) = \\ &= \sum_{\mu=1}^{l} Q_{\mu\nu}(\eta, \Xi_{-\eta}(x), \mathrm{pr}^{(n)} u(\Xi_{-\eta}(x))) \Delta_\mu(\Xi_{-\eta}(x), \mathrm{pr}^{(n)} u(\Xi_{-\eta}(x))). \end{aligned}$$

For any solution $U \in \mathcal{G}(\Omega)$ with representative $u = (u_\varepsilon)_{\varepsilon \in I}$, this expression is in $\mathcal{N}(\Omega)$ since $P_{\mu\nu} \in \mathcal{E}_M(\Omega)$ for each $\mu, \nu$, and every $\Delta_\mu(\Xi_{-\eta}(\,.\,), \mathrm{pr}^{(n)} u(\Xi_{-\eta}(\,.\,)))$ is in $\mathcal{N}(\Omega)$ because $U$ is a solution and $\Xi_{-\eta}$ is a diffeomorphism. □

**3.6 Example** The system

$$\begin{array}{rcl} U_t + UU_x & = & 0 \\ V_t + UV_x & = & 0 \\ U\,|_{\{t=0\}} = U_o & , & V\,|_{\{t=0\}} = V_o \end{array} \tag{9}$$

may serve as a simplified model for a one-dimensional, elastic material of high density in a nearly plastic state. It was analyzed in [25], where solutions $U, V \in \mathcal{G}_{s,g}(\mathbb{R} \times [0, \infty))$, $U_o, V_o \in \mathcal{G}_{s,g}(\mathbb{R})$ were constructed and studied ($\mathcal{G}_{s,g}$ is a variant of the Colombeau algebra with global instead of local bounds). In the following we present some applications of the above results to this system (for a more detailed study, see [21]). For $U_o' \ge 0$ (9) has a unique solution $(U, V)$ in $\mathcal{G}_{s,g}(\mathbb{R} \times [0, \infty))$

with $\partial_x U \geq 0$. We consider solutions in $\mathcal{G}_{s,g}(\mathbb{R} \times [0,\infty))$ with initial data $U_o(x) = u_L + (u_R - u_L)H(x)$ and $V_o(x) = v_L + (v_R - v_L)H(x)$, where $H$ is a generalized Heaviside function with $H' \geq 0$, i.e. $H$ is a member of $\mathcal{G}_{s,g}(\mathbb{R})$ with a representative $(h_\varepsilon)_{\varepsilon \in I}$ coinciding with the classical Heaviside function $Y$ off the interval $[-\varepsilon, \varepsilon]$. For $u_L < u_R$ the solution $(U,V)$ is associated with the rarefaction wave

$$u(x,t) = \begin{cases} u_L, & x \leq u_L t \\ \frac{x}{t}, & u_L t \leq x \leq u_R t \\ u_R, & u_R t \leq x \end{cases} \tag{10}$$

$$v(x,t) = \begin{cases} v_L \,, & x \leq u_L t \\ \left(\frac{v_R - v_L}{u_R - u_L}\right)\frac{x}{t} + \left(\frac{v_L u_R - v_R u_L}{u_R - u_L}\right), & u_L t \leq x \leq u_R t \\ v_R \,, & u_R t \leq x \end{cases} \tag{11}$$

However, choosing different generalized Heaviside functions for modelling the initial data $U_o$, respectively $V_o$ we may obtain a superposition of the rarefaction wave (10) in $u$ with a shock wave

$$v(x,t) = v_L + (v_R - v_L)Y(x - ct) \tag{12}$$

with arbitrary shock speed $c$, $u_L \leq c \leq u_R$. We are going to construct a one parameter symmetry $\eta \to g_\eta$ of (9) which transforms any of the solutions (11), (12) into a shock wave solution as $\eta \to \pm\infty$. For this we employ the two-dimensional Lorentz-transformation $(\eta, (x,t)) \to (x\cosh(\eta) - t\sinh(\eta), -x\sinh(\eta) + t\cosh(\eta))$ with infinitesimal generator $X_o = -t\partial_x - x\partial_t$. Then $X := X_o + (u^2-1)\partial_u$ generates a projectable one-parameter symmetry group of (9). Assuming that $-1 < u_{\mathrm{L}} < u_{\mathrm{R}} < 1$, we can extend the solution $(U,V)$ to the region $\Omega = \mathbb{R}^2 \setminus \{(x,t) : t \leq 0, u_{\mathrm{R}} t \leq x \leq u_{\mathrm{L}} t\}$ by the method of characteristics applied to representatives. Then the Lorentz-transformed solutions

$$\begin{aligned} \tilde{u}_\varepsilon(x,t) &= -\tanh(\eta - \operatorname{Artanh}(u_\varepsilon(x\cosh(\eta) + t\sinh(\eta), \\ &\qquad x\sinh(\eta) + t\cosh(\eta)))) \qquad (13) \\ \tilde{v}_\varepsilon(x,t) &= v_\varepsilon(x\cosh(\eta) + t\sinh(\eta), x\sinh(\eta) + t\cosh(\eta)) \qquad (14) \end{aligned}$$

(with Artanh the inverse of tanh) are well defined at least on $\mathbb{R} \times (0,\infty)$. The factorization property (5) in this case reads

$$\begin{aligned} &(\partial_t \tilde{u}_\varepsilon + \tilde{u}_\varepsilon \partial_x \tilde{u}_\varepsilon)(x,t) = \\ &\big((\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon)/(\cosh^3(\operatorname{Artanh}(u_\varepsilon - \eta))\cosh(\operatorname{Artanh}(u_\varepsilon)))\big)\left(\Xi_\eta^{-1}(x,t)\right) \end{aligned} \tag{15}$$

and similarly for the second line in (9), demonstrating that $(\widetilde{U}, \widetilde{V})$ is again a solution. For each $\eta$, $\widetilde{U}$ is associated with a piecewise smooth function which converges to $\mp 1$ as $\eta \to \pm\infty$. Observing that the coordinate transformations in (13), (14) approach boosts in the directions $(\mp 1, 1)$ as $\eta \to \pm\infty$, we see that the functions associated with $\widetilde{V}$ converge to the shock wave $v_L + (v_R - v_L)Y(x \pm t)$ as $\eta \to \pm\infty$, for whatever solution $V$ given in (11) or (12).

Although Proposition 3.5 provides a manageable algorithm to determine if classical symmetry groups carry over to generalized solutions it would certainly be preferable to have criteria at hand that allow to judge directly from the given PDE if the factors $P_{\mu\nu}$ behave nicely (given slowly increasing group actions). The first step in this direction is gaining control over the behaviour of the map $z \to \mathrm{pr}^{(n)} g_\eta(z)$, defined on $\mathcal{M}^{(n)}$.

**3.7 Proposition** *If $\eta \to g_\eta$ is a (strictly) slowly increasing group action on $\mathcal{M}$ then $z \to \mathrm{pr}^{(n)} g_\eta(z)$ is (strictly) slowly increasing as well.*

**Proof.** Let $N := \dim(\mathcal{M}^{(n)})$. For $z = (z_1, \ldots, z_p, z_{p+1}, \ldots, z_q, \ldots, z_N) \in \mathcal{M}^{(n)}$ we choose some smooth function $h : \mathcal{X} \to \mathcal{U}$ satisfying $z = z_h(z_1, \ldots, z_p)$, with $z_h(x)$ as in (7). Then we set $x := (z_1, \ldots, z_p)$, $u = (z_{p+1}, \ldots, z_q)$, $\widetilde{x} = \Xi_\eta(x)$ and $\widetilde{u} = \Phi_\eta(x, u)$. By the definition of prolongued group actions we have to find estimates for every

$$A_s := ((\Phi_\eta \circ (id \times h)) \circ \Xi_{-\eta})^{(s)} (\widetilde{x}) \tag{16}$$

(where $(s)$ denotes the derivative of order $s$) in terms of $z$. The above formula contains the components of $\mathrm{pr}^{(n)} g(z)$ of order $s$ ($s \le n$). Note that the particular choice of $h$ has no influence on (16), i.e. $A_s$ depends exclusively on $z$. To compute $A_s$ explicitly we use the formula for higher order derivatives of composite functions (see [11]). Denoting by $\Upsilon_m$ the group of permutations of $\{1, \ldots, m\}$ we have:

$$A_s(r_1, \ldots, r_s) = \sum_{i=1}^{s} \sum_{\substack{k \in \mathbb{N}^i \\ |k|=s}} \sum_{\sigma \in \Upsilon_s} \frac{1}{i!k!} (\Phi_\eta \circ (id \times h))^{(i)}((\widetilde{x}))(t_1, \ldots, t_i), \tag{17}$$

where

$$t_1 = \Xi_{-\eta}^{(k_1)}(\widetilde{x})(r_{\sigma(1)}, \ldots, r_{\sigma(k_1)}),\ \ldots, t_i = \Xi_{-\eta}^{(k_i)}(\widetilde{x})(r_{\sigma(s-k_i+1)}, \ldots, r_{\sigma(s)}).$$

and

$$(((\Phi_\eta \circ (id \times h)))^{(i)} (x)(t_1, \ldots, t_i) = \sum_{j=1}^{i} \sum_{\substack{l \in \mathbb{N}^j \\ |l|=i}} \sum_{\tau \in \Upsilon_i} \frac{1}{j!l!} \Phi_\eta^{(j)}(x, u)(s_1, \ldots, s_j), \tag{18}$$

where

$$s_1 = (id \times h)^{(l_1)}(x)(t_{\tau(1)}, \ldots, t_{\tau(l_1)}),\ \ldots, s_j = (id \times h)^{(l_j)}(x)(t_{\tau(i-l_j+1)}, \ldots, t_{\tau(i)}).$$

Each $s_m$ consists of sums of products of certain $t_{\tau(k)}$ with certain $z_l$ and an analogous assertion holds for the $\Phi_\eta^{(j)}(x, u)(s_1, \ldots, s_j)$. Hence from (17) and (18) the result follows. □

Returning to our original task of finding a priori estimates for the factors $P_{\mu\nu}$, even with the aid of Proposition 3.7 we still need some information about the explicit

form of the $Q_{\mu\nu}$ to go on. In general this seems quite difficult to achieve. However, there is a large and important class of PDEs that allow a priori statements on the concrete form of the factorization. Namely, we are going to show that each scalar PDE in which at least $u$ or one of its derivatives appears as a single term with constant coefficient belongs to this class.
Consider a scalar PDE $\Delta(x, u^{(n)}) = 0$ together with a symmetry group $\eta \to g_\eta$. Then we have

$$\Delta(z) = 0 \quad \Rightarrow \quad \Delta(\mathrm{pr}^{(n)} g_\eta(z)) = 0$$

Set $F(z) := \Delta(z)$, $f(z) := \Delta(\mathrm{pr}^{(n)} g_\eta(z))$ and $N = \dim(\mathcal{M}^{(n)})$. Suppose that in a neighborhood of some $\bar{z}$ with $F(\bar{z}) = 0$ we have $\frac{\partial F}{\partial z_k} > 0$ for some $1 \le k \le N$. Then by the implicit function theorem, locally there exists a smooth function $\psi : \mathbb{R}^{N-1} \to \mathbb{R}$ such that in a suitable neighborhood of $\bar{z}$ we have

$$F(z) = 0 \quad \Leftrightarrow \quad z_k = \psi(z'),$$

where $z' = (z_1, \dots, \hat{z_k}, \dots, z_N)$ (meaning that the component $z_k$ is missing from $z'$). It follows that

$$F(z) = (z_k - \psi(z')) \int_0^1 \frac{\partial F}{\partial z_k}(z_1, \dots, z_{k-1}, \tau z_k + (1-\tau)\psi(z'), \dots, z_N)\, d\tau,$$

and on the other hand

$$f(z) = (z_k - \psi(z')) \int_0^1 \frac{\partial f}{\partial z_k}(z_1, \dots, z_{k-1}, \tau z_k + (1-\tau)\psi(z'), \dots, z_N)\, d\tau.$$

Thus in the said neighborhood we have

$$f(z) = F(z) \frac{\int_0^1 \frac{\partial f}{\partial z_k}(z_1, \dots, z_{k-1}, \tau z_k + (1-\tau)\psi(z'), \dots, z_N) d\tau}{\int_0^1 \frac{\partial F}{\partial z_k}(z_1, \dots, z_{k-1}, \tau z_k + (1-\tau)\psi(z'), \dots, z_N) d\tau} \tag{19}$$

provided the denominator of this expression is $\neq 0$. In particular, if for some constant $c \neq 0$ we have $\frac{\partial F}{\partial z_k} \equiv c$ in a neighborhood of $\bar{z}$ then (19) simplifies to

$$f(z) = \frac{1}{c} F(z) \int_0^1 \frac{\partial f}{\partial z_k}(z_1, \dots, z_{k-1}, \tau z_k + (1-\tau)\psi(z'), \dots, z_N) d\tau \tag{20}$$

After these preparations we can state

**3.8 Theorem** *Let $\eta \to g_\eta$ be a slowly increasing symmetry group of the equation $\Delta(x, u^{(n)}) = 0$. Set $N = dim(\mathcal{M}^{(n)})$ and suppose that $\frac{\partial \Delta}{\partial z_k} \equiv c \neq 0$ for some $p + 1 \le k \le N$. Then $\eta \to g_\eta$ is a symmetry group of $\Delta(x, u^{(n)}) = 0$ in $\mathcal{G}$.*

**Proof.** Without loss of generality we may assume $c = 1$. Using the above notations we have $F(z) = z_k - \psi(z')$, so (20) implies

$$f(z) = F(z) \int_0^1 \frac{\partial f}{\partial z_k}(z_1, \dots, z_{k-1}, \tau z_k + (1-\tau)(z_k - F(z)), \dots, z_N) d\tau =: F(z) Q(\eta, z).$$

From Proposition 3.7 we know that $z \to f(z)$ is slowly increasing in the $u^{(n)}$-variables (i.e. in those $z_i$ with $i > p$), uniformly in $x = (z_1, \ldots, z_p)$ on compact sets. Since $F$ is slowly increasing we infer that $Q(\eta, z_u(x)) \in \mathcal{E}_M(\Omega)$ for any $u \in \mathcal{E}_M(\Omega)$ (with $z_u$ as in (7)). Finally,

$$\Delta(x, \mathrm{pr}^{(n)}(g_\eta u)(x)) = \Delta(\Xi_{-\eta}(x), \mathrm{pr}^{(n)} u(\Xi_{-\eta}(x)))Q(\eta, \Xi_{-\eta}(x), \mathrm{pr}^{(n)} u(\Xi_{-\eta}(x))).$$

Since $\Xi_{-\eta}$ is a diffeomorphism, it follows that if $U = \mathrm{cl}[u]$ solves the equation, so does $g_\eta U$. □

As the proof shows, we can drop the assumption $p + 1 \leq k$ if we require the group action to be strictly slowly increasing. It is clear that many PDEs satisfy the requirements of Theorem 3.8. For example, in the Hopf equation $\Delta(x, t, u, u_x, u_t) = u_t + uu_x$ or $\Delta(z_1, \ldots, z_5) = z_5 + z_3 z_4$ one can take $k = 5$. Note however that not every symmetry group of this equation is automatically slowly increasing. Theorem 3.8 constitutes a useful tool for transferring classical symmetry groups to Colombeau algebras.

**3.9 Example** We consider the initial value problem for the nonlinear transport equation

$$\begin{aligned} &U_t + \lambda \cdot \nabla_x U = f(U) \\ &U \mid_{\{t=0\}} = U_o \end{aligned} \tag{21}$$

with $t \in \mathbb{R}, x, \lambda \in \mathbb{R}^n$. It has unique solutions in $\mathcal{G}(\mathbb{R}^{n+1})$, given $U_o \in \mathcal{G}(\mathbb{R}^n)$, provided $f \in \mathcal{O}_M$ is globally Lipschitz (see [24]). If in addition $f$ is bounded and the initial data are distributions with discrete support, say $U_0(x) = \sum_{i,j} a_{ij} \delta^{(i)}(x - \xi_j)$ with $\xi_j \in \mathbb{R}^n, i \in \mathbb{N}_0^n$, then the generalized solution is associated with the delta wave $v + w$ where

$$v(x,t) = \sum_{i,j} a_{ij} \delta^{(i)}(x - \lambda t - \xi_j) \tag{22}$$

and $w$ is the smooth solution to $w_t + \lambda \cdot \nabla_x w = f(w)$, $w(0) = 0$.
The vector field $X = cf(u)\partial_u$ generates an infinitesimal symmetry of (21) for arbitrary $c \in \mathbb{R}$. With $F(u) := \int du/f(u)$, the corresponding Lie point transformation is

$$(x, t, u) \to (\widetilde{x}, \widetilde{t}, \widetilde{u}) = (x, t, F^{-1}\left(c\eta + F(u)\right)). \tag{23}$$

This provides a well-defined nonlinear transformation of the generalized solution $U \in \mathcal{G}(\mathbb{R}^{n+1})$, provided that the right hand side in (23) is slowly increasing.
In the example

$$U_t + \lambda \cdot \nabla_x U = \tanh(U) \tag{24}$$

the generalized solution is associated with $v(x,t)$ and $w$ vanishes identically. Applying (23) we obtain (due to Theorem 3.8) the new generalized solution

$$\widetilde{U}(x,t) = \mathrm{Arsinh}\left(e^{c\eta} \sinh(U(x,t))\right) \tag{25}$$

(with Arsinh the inverse of sinh). We are going to show that $\widetilde{U}$ is still associated with the delta wave $v$ in (22). To simplify the argument we assume $n = 1, \lambda = 0$ and $U_0(x) = \delta^{(i)}(x)$. Representatives of $U$ resp. $\widetilde{U}$ are $u_\varepsilon(x,t) = \mathrm{Arsinh}(e^t \sinh(\rho_\varepsilon^{(i)}(x)))$ and $\widetilde{u}_\varepsilon(x,t) = \mathrm{Arsinh}(e^{c\eta+t}\sinh(\rho_\varepsilon^{(i)}(x)))$. For $\psi \in \mathcal{D}(\mathbb{R}^2)$ we have

$$\begin{gathered} I_\varepsilon^i := \int\int \widetilde{u}_\varepsilon(x,t)\psi(x,t)dxdt = \\ = \int\int\int_0^1 \theta(e^{c\eta+t}, \sigma\varepsilon^{-i-1}\rho^{(i)}(x))d\sigma\varepsilon^{-i}\rho^{(i)}(x)\psi(\varepsilon x,t)dxdt \end{gathered}$$

where $\theta(\alpha, y) := \frac{d}{dy}\mathrm{Arsinh}(\alpha\sinh(y))$ for $\alpha > 0,\ y \in \mathbb{R}$. Since $\theta$ is bounded by $\max(1,\alpha)$ and $\lim_{|y|\to\infty}\theta(\alpha,y) = 1$ it follows that $I_\varepsilon^0 \to \int \psi(0,t)dt$, so $\widetilde{U}$ is associated with the delta function on the $t$-axis, as desired. For $i \geq 1$ we write

$$\begin{gathered} I_\varepsilon^i = \int\int\int_0^1 (\theta(e^{c\eta+t}, \sigma\varepsilon^{-i-1}\rho^{(i)}(x)) - 1)d\sigma\varepsilon^{-i}\rho^{(i)}(x)\psi(\varepsilon x,t)dxdt + \\ +(-1)^i \int\int \rho(x)\partial_x^i\psi(\varepsilon x,t)dxdt \end{gathered}$$

Here the second term converges to $(-1)^i \int \partial_x^i \psi(0,t)$ and the first term goes to zero since $\int_0^1 |\theta(\alpha,\sigma y) - 1|d\sigma \leq \frac{2|\alpha^2-1|}{\alpha|y|}(1 - e^{-|y|})$ for $y \neq 0$. This proves the claim for $\rho \in \mathcal{D}(\mathbb{R})$. For $\rho \in \mathcal{S}(\mathbb{R})$ splitting the $x$-integral into one from $-\frac{1}{\sqrt{\varepsilon}}$ to $\frac{1}{\sqrt{\varepsilon}}$ and one over $|x| \geq \frac{1}{\sqrt{\varepsilon}}$ gives the same result.

## 3.2 Continuity Properties

In this section we work out a different strategy for transferring classical point symmetries into the $\mathcal{G}$-setting. This approach, suggested in [28], consists in a more topological way of looking at the transfer problem by using continuity properties of differential operators. As we have pointed out in the discussion following (3), the main obstacle against directly applying classical symmetry groups componentwise to representatives of generalized solutions is that the differential equations need not be satisfied componentwise. However, there are certain classes of partial differential operators that do allow such a direct application. Consider a linear partial differential operator $P$ giving rise to an equation

$$PU = 0 \tag{26}$$

in $\mathcal{G}$ and let $G$ be a classical slowly increasing symmetry group of (26). Furthermore, suppose that $P$ possesses a continuous homogeneous (but not necessarily linear) right inverse $Q$. If $U = \mathrm{cl}[u]$ is a solution to (26) in $\mathcal{G}(\Omega)$ then there exists some $n \in \mathcal{N}(\Omega)$ such that

$$Pu = n.$$

Since $Q$ is a right inverse of $P$ this implies

$$P(u_\varepsilon - Qn_\varepsilon) = 0 \quad \forall\varepsilon \in I. \tag{27}$$

Also, $Qn \in \mathcal{N}(\Omega)$ due to the continuity and homogeneity assumption on $Q$. If $g \in G$, (27) implies

$$P(g(u_\varepsilon - Qn_\varepsilon)) = 0 \quad \forall \varepsilon \in I.$$

By definition,

$$P(gU) = \mathrm{cl}[P(gu)] = \mathrm{cl}[P(g(u - Qn))],$$

so $gU$ is a solution as well. Summing up, $G$ is a symmetry group in $\mathcal{G}$. The following result will serve to secure the existence of a right inverse as above for a large class of linear differential operators.

**3.10 Proposition** *Let $E$, $F$ be Fréchet spaces and $A$ a continuous linear map from $E$ onto $F$. Then $A$ has a continuous homogeneous right inverse $B : F \to E$.*

**Proof.** See [23], p. 364. □

From these preparations we conclude

**3.11 Theorem** *Let*

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \dots, l$$

*be a system of linear PDEs with slowly increasing $\Delta_\nu$ and let $\eta \to g_\eta$ be a slowly increasing symmetry group of this system. Assume that the operator defined by the left hand side is surjective $(\mathcal{C}^\infty(\Omega))^l \to (\mathcal{C}^\infty(\Omega))^l$. Then $\eta \to g_\eta$ is a symmetry group for the system in $\mathcal{G}(\Omega)$ as well.* □

The assumptions of Theorem 3.11 are automatically satisfied for any linear partial differential operator with constant coefficients on an arbitrary convex open domain (see [17], 10.6).

**3.12 Example** The system of one-dimensional linear acoustics

$$\begin{aligned} P_t + U_x &= 0 \\ U_t + P_x &= 0. \end{aligned} \tag{28}$$

is transformed via $U = V - W, P = V + W$ into

$$\begin{aligned} V_t + V_x &= 0 \\ W_t - W_x &= 0. \end{aligned} \tag{29}$$

Using the infinitesimal generators $\Phi(v)\partial_v + \Psi(w)\partial_w$ ($\Phi$, $\Psi$ arbitrary smooth functions) of (29) we obtain symmetry transformations for (28) of the form

$$\begin{aligned} \widetilde{U} &= F^{-1}\left(\eta + F(\frac{1}{2}(P + U))\right) - G^{-1}\left(\theta + G(\frac{1}{2}(P - U))\right) \\ \widetilde{P} &= F^{-1}\left(\eta + F(\frac{1}{2}(P + U))\right) + G^{-1}\left(\theta + G(\frac{1}{2}(P - U))\right) \end{aligned}$$

with arbitrary diffeomorphisms $F, G$. Since (28) satisfies the assumptions of Theorem 3.11 on $\Omega = \mathbb{R}^2$ it follows that any slowly increasing transformation of this form is a symmetry of (28). In particular, this includes nonlinear transformations of distributional solutions, cf. Example 3.13.

In the remainder of this section we discuss the interplay between symmetry groups and solutions of PDEs in the sense of association. Consider

$$\Delta_\nu(x, u^{(n)}) \approx 0, \quad 1 \le \nu \le l \tag{30}$$

in $\mathcal{G}$. A slowly increasing symmetry group of the corresponding system

$$\Delta(x, u^{(n)}) = 0, \quad 1 \le \nu \le l$$

is called a symmetry group in the sense of association if it transforms solutions of (30) into other such solutions. The first question to be answered in this context is whether one can derive conditions on the form of the factorization (8) that will yield symmetry groups in the sense of association. It is clear that a sufficient condition is to suppose that $Q$ depends exclusively on $\eta$ and $x$. Distributional solutions to linear PDEs arise as a special case of (30) and have been treated in [4]. There, the validity of equation (8) with $Q$ depending on $\eta$ and $x$ only is actually used to *define* symmetry groups in $\mathcal{D}'$. In order to remain within the classical distributional framework, the admissible group transformations in [4] are restricted to projectable ones acting linearly in the dependent variables. On the other hand, the method developed there is even applicable to linear equations containing distributional terms which allows to use invariance methods to compute fundamental solutions.
Second, if $u$ is a solution to $\Delta(x, u^{(n)}) = 0$ in $\mathcal{G}(\Omega)$ possessing an associated distribution, one may ask for which group actions $g$ this implies that $gu$ as well possesses an associated distribution. This is certainly the case for admissible transformations in the above sense. On the other hand, we have already seen in Example 3.9 that even genuinely nonlinear symmetry transformations may preserve association properties.
The next example shows that nonlinear group actions may transform distributional solutions in Examples 3.9 and 3.12 into more complicated distributional solutions or into generalized solutions in $\mathcal{G}(\mathbb{R}^2)$ not admitting associated distributions.

**3.13 Example** We consider the equation $U_t + \lambda U_x = 0$ arising in (21) with $n = 1$ or in (29). We have already observed that $\widetilde{U} = F^{-1}(\eta + F(U))$ defines a symmetry transformation for arbitrary diffeomorphisms $F$. Here we take $F \in \mathcal{C}^\infty(\mathbb{R})$, $F' > 0$, $F(y) = \mathrm{sign}(y)\sqrt{|y|}$ for $|y| \ge 1$. We wish to compute $\widetilde{U}$ when $U \in \mathcal{G}(\mathbb{R}^2)$ is a delta wave solution $U(x,t) \approx \delta^{(i)}(x - \lambda t)$. We take $U$ as the class of $\rho_\varepsilon^{(i)}(x - \lambda t)$ with $\rho \in \mathcal{D}([-1, 1])$. We have when $\eta \ge 0$:

(i) If $i = 0$, that is $U \approx \delta(x - \lambda t)$, then $\widetilde{U} \approx F^{-1}(\eta + F(0)) + \delta(x - \lambda t)$;

(ii) If $i = 1$, that is $U \approx \delta'(x - \lambda t)$, then
$\widetilde{U} \approx F^{-1}(\eta + F(0)) + 2\eta \int \sqrt{|\rho'(y)|}\, dy\ \delta(x - \lambda t) + \delta'(x - \lambda t)$;

(iii) If $i \ge 2$ then $\widetilde{U}$ does not admit an associated distribution.

To see this, we may assume that $\lambda = 0$ and write $a_\varepsilon(x) := \rho_\varepsilon^{(i)}(x)$ for brevity. Note that $F^{-1}(y) = \mathrm{sign}(y)y^2$ for $|y| \geq 1$. Let $A_\varepsilon = \{x \in [-\varepsilon, \varepsilon] : |a_\varepsilon(x)| \geq (\eta+1)^2\}$. If $x \in A_\varepsilon$ and $a_\varepsilon(x) \geq 0$ then $\eta + F(a_\varepsilon(x)) \geq 1$ and $F^{-1}(\eta + F(a_\varepsilon(x))) = \eta^2 + 2\eta\sqrt{a_\varepsilon(x)} + a_\varepsilon(x)$. Also, if $x \in A_\varepsilon$ and $a_\varepsilon(x) < 0$ then $\eta + F(a_\varepsilon(x)) \leq -1$ and $F^{-1}(\eta + F(a_\varepsilon(x))) = -\eta^2 + 2\eta\sqrt{|a_\varepsilon(x)|} + a_\varepsilon(x)$. The functions $F^{-1}(\eta + F(a_\varepsilon))$, $|a_\varepsilon(x)|$ and $\sqrt{|a_\varepsilon(x)|}$ are bounded on the complement of $A_\varepsilon$. Thus

$$\begin{aligned} &\textstyle\int\int_{-\varepsilon}^{\varepsilon} F^{-1}(\eta + F(a_\varepsilon(x)))\psi(x,t)dxdt = \\ &= \textstyle\int\int_{A_\varepsilon}(\pm\eta^2 + 2\eta\sqrt{|a_\varepsilon(x)|} + a_\varepsilon(x))\psi(x,t)dxdt + O(\varepsilon) = \\ &= \textstyle\int\int_{-\varepsilon}^{\varepsilon}(2\eta\sqrt{|a_\varepsilon(x)|} + a_\varepsilon(x))\psi(x,t)dxdt + O(\varepsilon) \end{aligned}$$

while

$$\int\int_{|x|\geq\varepsilon} F^{-1}(\eta + F(a_\varepsilon(x)))\psi(x,t)\,dxdt \to F^{-1}(\eta + F(0))\int\int \psi(x,t)\,dxdt$$

It follows that $F^{-1}(\eta + F(a_\varepsilon(x)))$ converges in $\mathcal{D}'(\mathbb{R}^2)$ if and only if $2\eta\sqrt{|a_\varepsilon|} + a_\varepsilon$ admits an associated distribution. A simple computation yields the particular results (i), (ii), (iii).

# 4 Generalized Group Actions

Although the methods introduced in the previous sections enable an application of large classes of classical symmetry groups to elements of Colombeau algebras, they are but the first step in a theory of generalized group analysis of differential equations. In this section we develop an extension of the methods of group analysis that will allow to consider symmetry groups of differential equations whose actions are generalized functions themselves.

## 4.1 Generalized Transformation Groups

Simple examples indicate the necessity of extending the methods of group analysis of PDEs to equations involving generalized functions themselves:

**4.1 Example** Considering (21) in $\mathcal{G}_\tau$ with a *generalized* function $f = \mathrm{cl}[(f_\varepsilon)_{\varepsilon\in I}] \in \mathcal{G}_\tau$ we can apply the classical algorithm for calculating symmetry groups componentwise to the equations

$$\partial_t u_\varepsilon + \lambda \cdot \nabla_x u_\varepsilon = f_\varepsilon(u_\varepsilon)$$

thereby obtaining infinitesimal generators with generalized coefficient functions. Thus the question arises in which sense such generators induce symmetries of the differential equation. More generally, one can consider differential equations in $\mathcal{G}_\tau$ of the form

$$P(x, U^{(n)}) = 0$$

where $P$ is a generalized function.

As is indicated by Example 4.1, composition of generalized functions will inevitably occur in a generalization of group analysis. For this purpose, we shall apply suitable variants of Colombeau algebras for the following considerations, namely $\mathcal{G}_\tau(\mathbb{R}^n)$ and $\widetilde{\mathcal{G}}_\tau(\mathbb{R}\times\mathbb{R}^n) = \widetilde{\mathcal{G}}_\tau(\mathbb{R}^{1+n})$.

**4.2 Definition** *A generalized group action on* $\mathbb{R}^n$ *is an element* $\Phi$ *of* $(\widetilde{\mathcal{G}}_\tau(\mathbb{R}^{1+n}))^n$ *such that:*

(i) $\Phi(0,\,.\,) = \mathrm{id}$ in $(\mathcal{G}_\tau(\mathbb{R}^n))^n$.

(ii) $\Phi(\eta_1+\eta_2,\,.\,) = \Phi(\eta_1, \Phi(\eta_2,\,.\,))$ in $(\mathcal{G}_\tau(\mathbb{R}^{2+n}))^n$.

Before we turn to an infinitesimal description of generalized group actions let us shortly recall some basic definitions from [27] that are needed for a pointvalue characterization of generalized functions which in turn plays a fundamental role in the following considerations. Thus for any open set $\Omega \subseteq \mathbb{R}^n$ we set

$$\Omega_M := \{(x_\varepsilon)_{\varepsilon\in I} \in \Omega^I : \exists p>0\ \exists\ \eta>0\ |x_\varepsilon| \le \varepsilon^{-p}\ (0<\varepsilon<\eta)\}.$$

On $\Omega_M$ we define an equivalence relation by

$$(x_\varepsilon)_{\varepsilon\in I} \sim (y_\varepsilon)_{\varepsilon\in I} \ \Leftrightarrow\ \forall q>0\ \exists\eta>0\ |x_\varepsilon - y_\varepsilon| \le \varepsilon^q\ (0<\varepsilon<\eta)$$

and set $\widetilde{\Omega} := \Omega_M/\sim$. $\widetilde{\Omega}$ is called the set of generalized points corresponding to $\Omega$. The set of compactly supported points is defined as

$$\widetilde{\Omega}_c = \{\widetilde{x} \in \widetilde{\Omega} : \exists \text{ representative } (x_\varepsilon)_{\varepsilon\in I}\ \exists K \subset\subset \Omega\ \exists\eta>0 : \ x_\varepsilon \in K,\ \varepsilon\in(0,\eta)\}.$$

Note that for $\Omega=\mathbb{R}$ we have $\widetilde{\Omega} = \mathcal{R}$. Theorems 2.4, 2.7 and 2.10 of [27] establish that elements of $\mathcal{G}(\Omega)$, $\widetilde{\mathcal{G}}_\tau(\Omega)$ or $\widetilde{\mathcal{G}}_\tau(\Omega\times\Omega')$ are uniquely determined by their pointvalues in $\widetilde{\Omega}_c$, $\widetilde{\Omega}$, or $\widetilde{\Omega}_c \times \widetilde{\Omega}'$, respectively. For the theory of ODEs in the Colombeau framework we refer to [16].

**4.3 Definition** *Let* $\xi = (\xi_1,\dots,\xi_n) \in (\mathcal{G}_\tau(\mathbb{R}^n))^n$. *The generalized vector field* $X = \sum_{i=1}^n \xi_i(x)\partial_{x_i}$ *is called* $\mathcal{G}$*-complete if the initial value problem*

$$\begin{aligned} \dot{x}(t) &= \xi(x(t)) \\ x(t_o) &= \widetilde{x}_o \end{aligned}$$

*is uniquely solvable in* $\mathcal{G}(\mathbb{R})^n$ *for any* $\widetilde{x}_o \in \mathcal{R}^n$ *and any* $t_o \in \mathbb{R}$.

**4.4 Definition** *Let* $\Phi$ *be a generalized group action on* $\mathbb{R}^n$ *and set*

$$\xi := \frac{d}{d\eta}\Big|_0 \Phi(\eta,\,.\,) \in (\mathcal{G}_\tau(\mathbb{R}^n))^n.$$

*If the generalized vector field* $X = \sum_{i=1}^n \xi_i(x)\partial_{x_i}$ *is* $\mathcal{G}$*-complete, then* $X$ *is called the infinitesimal generator of* $\Phi$. *In this case,* $\Phi$ *is also called* $\mathcal{G}$*-complete.*

By [16], every generalized vector field with $\mathcal{G}_\tau$-components whose gradient is of $L^\infty$-log-type is $\mathcal{G}$-complete. The notion of infinitesimal generator is well-defined due to

**4.5 Proposition** *Every $\mathcal{G}$-complete generalized group action is uniquely determined by its infinitesimal generator.*

**Proof.** Let $\Phi'$, $\Phi''$ be two $\mathcal{G}$-complete generalized group actions with the same infinitesimal generator $X = \sum_{i=1}^n \xi_i(x)\partial_{x_i}$. Then both functions satisfy

$$\frac{d}{d\eta}\Phi(\eta,x) = \frac{d}{d\mu}\Big|_0 \Phi(\eta+\mu,x) = \frac{d}{d\mu}\Big|_0 \Phi(\mu,\Phi(\eta,x)) = \xi(\Phi(\eta,x)).$$

Now given any $\widetilde{x} \in \mathcal{R}^n$, it follows that both $\eta \to \Phi'(\eta,\widetilde{x})$ and $\eta \to \Phi''(\eta,\widetilde{x})$ solve the initial value problem

$$\begin{aligned} \dot{x}(\eta) &= \xi(x(\eta)) \\ x(0) &= \widetilde{x} \end{aligned}$$

By assumption this entails that $\Phi'(\,.\,,\widetilde{x}) = \Phi''(\,.\,,\widetilde{x})$ in $(\mathcal{G}(\mathbb{R}))^n$. Consequently,

$$\Phi'(\widetilde{\eta},\widetilde{x}) = \Phi''(\widetilde{\eta},\widetilde{x})$$

for all $\widetilde{\eta} \in \mathcal{R}_c$ and all $\widetilde{x} \in \mathcal{R}^n$. The claim now follows from [27], Theorem 2.10. $\square$

As in the classical theory, we are first going to investigate symmetry groups of algebraic equations:

**4.6 Definition** *Let $F \in \mathcal{G}_\tau(\mathbb{R}^n)$ and let $\Phi$ be a generalized group action on $\mathbb{R}^n$. $\Phi$ is called a symmetry group of the equation*

$$F(x) = 0$$

*in $\mathcal{G}_\tau(\mathbb{R}^n)$ if for any $\widetilde{x} \in \mathcal{R}^n$ with $F(\widetilde{x}) = 0 \in \mathcal{R}$ it follows that $\eta \to F(\Phi(\eta,\widetilde{x})) = 0$ in $\mathcal{G}(\mathbb{R})$ (or, equivalently, $F(\Phi(\widetilde{\eta},\widetilde{x})) = 0$ in $\mathcal{R}$ for every $\widetilde{\eta} \in \mathcal{R}_c$).*

A characterization of symmetry groups of (generalized) algebraic equations in terms of infinitesimal generators is provided by

**4.7 Theorem** *Let $F \in \mathcal{G}_\tau(\mathbb{R}^n)$ be of the form*

$$F(x_1,\dots,x_n) = x_i - f(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$$

*for some $1 \le i \le n$ and $f \in \mathcal{G}_\tau(\mathbb{R}^{n-1})$. Let $\Phi$ be a $\mathcal{G}$-complete generalized group action with infinitesimal generator $X = \sum_{i=1}^n \xi_i(x)\partial_{x_i}$ and suppose that $x' \to \xi(x',f(x'))$ defines a generalized vector field on $\mathbb{R}^{n-1}$ such that the corresponding system of ODEs possesses a flow in $(\widetilde{\mathcal{G}}_\tau(\mathbb{R}^{1+(n-1)}))^{n-1}$. The following conditions are equivalent:*

*(i)* $\Phi$ *is a symmetry group of* $F(x) = 0$.

*(ii) If* $\widetilde{x} \in \mathcal{R}^n$ *with* $F(\widetilde{x}) = 0 \in \mathcal{R}$ *it follows that* $X(F)(\widetilde{x}) = 0$ *in* $\mathcal{R}$.

**Proof.** (i) $\Rightarrow$ (ii): Consider the function $(\eta, x) \to F(\Phi(\eta, x)) \in \widetilde{\mathcal{G}}_\tau(\mathbb{R}^{1+n})$. We have

$$\frac{d}{d\eta} F(\Phi(\eta, x)) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\Phi(\eta, x))\xi_i(\Phi(\eta, x)) = X(F)(\Phi(\eta, x)),$$

so that $\frac{d}{d\eta}\big|_0 F(\Phi(\eta, x)) = X(F)(x)$ in $\mathcal{G}_\tau(\mathbb{R}^n)$. Let $\widetilde{x} \in \mathcal{R}^n$ such that $F(\widetilde{x}) = 0$. Then $F(\Phi(\,.\,, \widetilde{x})) = 0$ in $\mathcal{G}(\mathbb{R})$. Thus $\frac{d}{d\eta}\big|_0 F(\Phi(\eta, \widetilde{x})) = 0$ in $\mathcal{R}$ which means that $X(F)(\widetilde{x}) = 0$ in $\mathcal{R}$.
(ii) $\Rightarrow$ (i): We assume $F(x_1, \ldots, x_n) = x_n - f(x_1, \ldots, x_{n-1})$ and abbreviate ($x_1$, $\ldots$, $x_{n-1}$) by $x'$. Our first claim is that

$$\xi_n(x', f(x')) = \sum_{j=1}^{n-1} \xi_j(x', f(x'))\partial_j f(x') \text{ in } \mathcal{G}_\tau(\mathbb{R}^{n-1})$$

Indeed, if $\widetilde{x}' \in \mathcal{R}^{n-1}$ then $F(\widetilde{x}', f(\widetilde{x}')) = 0$ in $\mathcal{R}$. Hence $X(F)(\widetilde{x}', f(\widetilde{x}')) = 0$ in $\mathcal{R}$ for all $\widetilde{x}'$ by our assumption. Our claim now follows from [27], Theorem 2.7. Consider the following system of ODEs in $\mathcal{G}_\tau$:

$$\begin{aligned} \dot{x}_j(t) &= \xi_j(x', f(x')) \quad (j = 1, \ldots, n-1) \\ x'(0) &= \widetilde{a}' \in \mathcal{R}^{n-1} \end{aligned}$$

By our assumption, this system has a flow $(\eta, a') \to (h_1(\eta, a'), \ldots, h_{n-1}(\eta, a'))$ in $(\widetilde{\mathcal{G}}_\tau(\mathbb{R}^{1+(n-1)}))^{n-1}$. Set $g_n(\eta, a) := f(h_1(\eta, a'), \ldots, h_{n-1}(\eta, a'))$. Then $g_n(0, a) = f(a')$ and

$$g(\eta, a) = (g_1(\eta, a), \ldots, g_n(\eta, a)) := (h_1(\eta, a'), \ldots, h_{n-1}(\eta, a'), g_n(\eta, a))$$

is in $(\widetilde{\mathcal{G}}_\tau(\mathbb{R}^{1+n}))^n$. If $\widetilde{a} \in \mathcal{R}^n$ then $F(g(\eta, \widetilde{a})) = 0$ in $\mathcal{R}$ for all $\eta \in \mathcal{R}_c$. Therefore, if we can show that $g(\,.\,, \widetilde{a}) = \Phi(\,.\,, \widetilde{a})$ in $(\mathcal{G}(\mathbb{R}))^n$ for all $\widetilde{a}$ with $F(\widetilde{a}) = 0$, the proof is completed. Now we have $\dot{g}_j(\eta, a) = \xi_j(g_1(\eta, a), \ldots, g_n(\eta, a))$ for $1 \leq j \leq n-1$ and

$$\begin{aligned} \dot{g}_n(\eta, a) = \textstyle\sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(g_1(\eta, a), \ldots, g_{n-1}(\eta, a))\dot{g}_i(\eta, a) = \\ = \xi_n(g_1(\eta, a), \ldots, f(g_1(\eta, a), \ldots, g_{n-1}(\eta, a))) = \xi_n(g(\eta, a)). \end{aligned}$$

If $F(\widetilde{a}) = 0$ in $\mathcal{R}$ then $\widetilde{a}_n = f(\widetilde{a}')$, so that $g(0, \widetilde{a}) = (\widetilde{a}', f(\widetilde{a}')) = \widetilde{a} = \Phi(0, \widetilde{a})$. Thus $g(\,.\,, \widetilde{a})$ and $\Phi(\,.\,, \widetilde{a})$ solve the same initial value problem. Since $X$ is $\mathcal{G}$-complete, the claim follows. $\square$

## 4.2 Symmetries of Differential Equations

In this section we are going to apply the above results to symmetry groups of differential equations involving generalized functions. To this end, we will first have to define generalized group actions on generalized functions. Once we have done this, by a symmetry group of a differential equation we will again mean a group action that transforms solutions into other solutions. Thus, from now on we will exclusively consider group actions on some space $\mathbb{R}^p \times \mathbb{R}^q$ of independent and dependent variables.

**4.8 Definition** *A generalized group action $\Phi \in (\widetilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^{p+q}))^{p+q}$ is called projectable if it is of the form*

$$\Phi(\eta, (x, u)) = (\Xi_\eta(x), \Psi_\eta(x, u)),$$

*where $\Xi \in (\widetilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^p))^p$ and $\Psi \in (\widetilde{\mathcal{G}}_\tau(\mathbb{R} \times \mathbb{R}^{p+q}))^q$.*

The group properties in this case read:

$$\Xi_{\eta_1+\eta_2} = \Xi_{\eta_1} \circ \Xi_{\eta_2} \quad \text{in } \mathcal{G}_\tau(\mathbb{R}^p)\ \forall \eta_1, \eta_2 \in \mathcal{R}_c. \tag{31}$$

$$\Psi_{\eta_1+\eta_2}(x, u) = \Psi_{\eta_1}(\Xi_{\eta_2}(x), \Psi_{\eta_2}(x, u)) \quad \text{in } \mathcal{G}_\tau(\mathbb{R}^{p+q})\ \forall \eta_1, \eta_2 \in \mathcal{R}_c. \tag{32}$$

In particular, we have

$$\Xi_\eta \circ \Xi_{-\eta} = \text{id} \quad \text{in } \mathcal{G}_\tau(\mathbb{R}^p)\ \forall \eta \in \mathcal{R}_c. \tag{33}$$

An adaptation of Lie group analysis to spaces of distributions faces the fundamental problem that while the methods of classical Lie group analysis of differential equations are *geometric* in the sense that group action on functions is defined via graphs, in classical distribution theory there is no means of defining graphs of distributions. However, due to the pointvalue characterization obtained in [27] this problem can be dealt with in a satisfactory manner within Colombeau algebras:

**4.9 Definition** *Let $U \in (\mathcal{G}(\mathbb{R}^p))^q$ and $V \in (\mathcal{G}_\tau(\mathbb{R}^p))^q$. The graphs of $U$ and $V$ are defined as*

$$\Gamma_U := \{(\widetilde{x}, U(\widetilde{x})) : \widetilde{x} \in \mathcal{R}_c^p\}$$
$$\Gamma_V := \{(\widetilde{x}, U(\widetilde{x})) : \widetilde{x} \in \mathcal{R}^p\}.$$

It follows directly from [27], Theorems 2.4 and 2.7 that any generalized function is uniquely determined by its graph. Our next aim is to define generalized group actions on generalized functions. As in the classical case this is done geometrically, i.e. by transformation of graphs. The following result is immediate from the definitions:

**4.10 Proposition** *Let $U \in (\mathcal{G}_\tau(\mathbb{R}^p))^q$ and let $\Phi$ be a projectable generalized group action on $\mathbb{R}^p \times \mathbb{R}^q$. Then $\Phi_\eta(\Gamma_U) = \Gamma_{\Phi_\eta(U)}$ in $\mathcal{R}^{p+q}$ for each $\eta$, where $\Phi_\eta(U)$ denotes the element*

$$x \to \Psi_\eta(\Xi_{-\eta}(x), U \circ \Xi_{-\eta}(x))$$

*of $(\mathcal{G}_\tau(\mathbb{R}^p))^q$.* □

We are now able to give a geometric characterization of solutions of PDEs in $\mathcal{G}_\tau$.

**4.11 Proposition** *Consider the system of PDEs*

$$\Delta_\nu(x, U^{(n)}) = 0 \quad 1 \le \nu \le l \tag{34}$$

*in $\mathcal{G}_\tau(\mathbb{R}^p))^q$ (where $\Delta \in (\mathcal{G}_\tau((\mathbb{R}^p \times \mathbb{R}^q)^{(n)}))^l$). Set*

$$\mathcal{S}_\Delta := \{\widetilde{z} \in \mathcal{R}^{(n)} \,:\, \Delta_\nu(\widetilde{z}) = 0 \ (1 \le \nu \le l)\}.$$

*Then $U \in (\mathcal{G}_\tau(\mathbb{R}^p))^q$ is a solution of the system iff $\Gamma_{\mathrm{pr}^{(n)}U} \subseteq \mathcal{S}_\Delta$.*

**Proof.** This follows immediately from [27], Theorem 2.7. □

Prolongation of generalized group actions can be handled in a similar fashion as in the classical theory. Thus, let $\Phi$ be a projectable generalized group action on $\mathbb{R}^p \times \mathbb{R}^q$. We want to define the $n$-th prolongation $\mathrm{pr}^{(n)}\Phi$ as a projectable generalized group action on $(\mathbb{R}^p \times \mathbb{R}^q)^{(n)}$. Let $z \in (\mathbb{R}^p \times \mathbb{R}^q)^{(n)}$ and choose $h \in \mathcal{O}_M(\mathbb{R}^p)^q$ such that $(z_1, \dots, z_p, \mathrm{pr}^{(n)}h(z_1, \dots, z_p)) = z$. Now set

$$\mathrm{pr}^{(n)}\Phi(\eta, z) := (\Xi_\eta(z_1, ..., z_p), \mathrm{pr}^{(n)}(\Phi_\eta(h))(\Xi_\eta(z_1, ..., z_p))). \tag{35}$$

Using for $h$ a suitable Taylor polynomial, it follows that $\mathrm{pr}^{(n)}\Phi \in (\widetilde{\mathcal{G}}_\tau(\mathbb{R} \times (\mathbb{R}^p \times \mathbb{R}^q)^{(n)})^N$ (where $N = \dim((\mathbb{R}^{p+q})^{(n)})$). Moreover, the definition does not depend on the particular choice of $h$, which follows exactly as in the classical case.

**4.12 Lemma** *Let $\widetilde{z} \in (\mathcal{R}^p \times \mathcal{R}^q)^{(n)}$ and assume that $U \in (\mathcal{G}_\tau(\mathbb{R}^p))^q$ satisfies $(\widetilde{z}_1, \dots, \widetilde{z}_p, \mathrm{pr}^{(n)}U(\widetilde{z}_1, \dots, \widetilde{z}_p)) = \widetilde{z}$. Then*

$$\mathrm{pr}^{(n)}\Phi(\eta, \widetilde{z}) = (\Xi_\eta(\widetilde{z}_1, \dots, \widetilde{z}_p), \mathrm{pr}^{(n)}(\Phi_\eta(U))(\Xi_\eta(\widetilde{z}_1, \dots, \widetilde{z}_p))) \quad \forall \eta \in \mathcal{R}_c. \tag{36}$$

**Proof.** Let $U = \mathrm{cl}[(u_\varepsilon)_{\varepsilon \in I}]$ and choose a representative $(z_\varepsilon)_{\varepsilon \in I}$ of $\widetilde{z}$ such that

$$(z_{1\varepsilon}, \dots, z_{p\varepsilon}, \mathrm{pr}^{(n)}u_\varepsilon(z_{1\varepsilon}, \dots, z_{p\varepsilon})) = z_\varepsilon \quad \forall \varepsilon.$$

Using the chain rule as in Proposition 3.7, it follows that the right hand sides of (35) (with $z$ replaced by $\widetilde{z}$) and of (36) have the same representative (depending *exclusively* on $(z_\varepsilon)_{\varepsilon \in I}$). □

**4.13 Proposition** $\mathrm{pr}^{(n)}\Phi$ *is a generalized group action on $(\mathbb{R}^p \times \mathbb{R}^q)^{(n)}$.*

**Proof.** Property 4.2 (i) is clearly satisfied. Concerning (ii), according to [27], Theorem 2.7 it suffices to show that

$$\mathrm{pr}^{(n)}\Phi(\eta_1+\eta_2,\widetilde{z}) = \mathrm{pr}^{(n)}\Phi(\eta_1,\mathrm{pr}^{(n)}\Phi(\eta_2,\widetilde{z})) \quad \forall \eta_1,\eta_2 \in \mathcal{R}_c,\ \forall \widetilde{z} \in (\mathcal{R}^p \times \mathcal{R}^q)^{(n)}.$$

Choose some $U \in (\mathcal{G}_\tau(\mathbb{R}^p))^q$ with $(\widetilde{z}_1,\dots,\widetilde{z}_p,\mathrm{pr}^{(n)}U(\widetilde{z}_1,\dots,\widetilde{z}_p)) = \widetilde{z}$. Then due to Lemma 4.12 we have

$$\mathrm{pr}^{(n)}\Phi(\eta_2,\widetilde{z}) = (\Xi_{\eta_2}(\widetilde{z}_1,\dots,\widetilde{z}_p),\mathrm{pr}^{(n)}(\Phi_{\eta_2}(U))(\Xi_{\eta_2}(\widetilde{z}_1,\dots,\widetilde{z}_p))).$$

By (36) this implies $\mathrm{pr}^{(n)}\Phi(\eta_1,\mathrm{pr}^{(n)}\Phi(\eta_2,\widetilde{z})) = \mathrm{pr}^{(n)}\Phi(\eta_1+\eta_2,\widetilde{z})$. □

As in the classical case we therefore have (using the notations from Proposition 4.11):

**4.14 Proposition** *Let $\Phi$ be a projectable generalized group action on $\mathbb{R}^p \times \mathbb{R}^q$ such that $\mathrm{pr}^{(n)}\Phi$ is a symmetry group of the algebraic equation $\Delta(z)=0$. Then $\Phi$ is a symmetry group of (34).*

**Proof.** If $U \in \mathcal{G}_\tau(\mathbb{R}^p)$ is a solution of (34) then $\Gamma_{\mathrm{pr}^{(n)}U} \subseteq \mathcal{S}_\Delta$ by Proposition 4.11. Thus

$$\Gamma_{\mathrm{pr}^{(n)}(\Phi_\eta U)} = \mathrm{pr}^{(n)}\Phi_\eta(\Gamma_{\mathrm{pr}^{(n)}U}) \subseteq \mathcal{S}_\Delta,$$

so that, again from Proposition 4.11, the claim follows. □

**4.15 Definition** *Let $X$ be a $\mathcal{G}$-complete generalized vector field. The $n$-th prolongation of $X$ is defined as the infinitesimal generator of the $n$-th prolongation of the generalized group action $\Phi$ corresponding to $X$:*

$$\mathrm{pr}^{(n)}X|_z = \frac{d}{d\eta}\Big|_0 \mathrm{pr}^{(n)}\Phi_\eta(z),$$

*provided that $\mathrm{pr}^{(n)}\Phi$ is $\mathcal{G}$-complete as well. In this case, both $X$ and $\Phi$ are called $\mathcal{G}$-$n$-complete.*

From Theorem 4.7 and Proposition 4.14 we immediately conclude

**4.16 Theorem** *Under the assumptions of Proposition 4.11, let $\Phi$ be a $\mathcal{G}$-$n$-complete generalized group action on $\mathbb{R}^p \times \mathbb{R}^q$ with infinitesimal generator $X$ such that the conditions of Theorem 4.7 are satisfied for $\Delta$ and $\mathrm{pr}^{(n)}\Phi$. If*

$$\mathrm{pr}^{(n)}X(\Delta)(\widetilde{z}) = 0 \quad \forall \widetilde{z} \in (\mathcal{R}^p \times \mathcal{R}^q)^{(n)} \text{ with } \Delta(\widetilde{z}) = 0,$$

*then $\Phi$ is a symmetry group of (34).* □

In order to be able to apply the same algorithm as in classical Lie theory for the determination of the symmetry group of a generalized PDE, the final step is to verify that the formulas for prolongation of vector fields carry over to generalized vector fields.

**4.17 Theorem** *Let*

$$X = (x,u) \to \sum_{i=1}^{p} \xi_i(x)\partial_{x_i} + \sum_{\alpha=1}^{q} \psi_\alpha(x,u)\partial_{u^\alpha}$$

*be a $\mathcal{G}$-$n$-complete generalized vector field with corresponding projectable group action $\Phi$ on $(\mathbb{R}^p \times \mathbb{R}^q)$. Then*

$$\mathrm{pr}^{(n)} X = X + \sum_{\alpha=1}^{q} \sum_{J} \psi_\alpha^J(x, u^{(n)})\partial_{u_J^\alpha}$$

*where $J = (j_1, ..., j_k)$, $1 \le j_k \le p$ for $1 \le k \le n$ and*

$$\psi_\alpha^J(x, u^{(n)}) = D_J(\psi_\alpha - \sum_{i=1}^{p} \xi_i u_i^\alpha) + \sum_{i=1}^{p} \xi_i u_{J,i}^\alpha$$

**Proof.** Using the machinery developed so far, this is an easy modification of the proof of the classical result (see [29], Theorem 2.36). □

We may summarize the results of this section as follows: In order to determine the symmetries of a differential equation involving generalized functions, the algorithm (as in the classical case) is to make an ansatz for the infinitesimal generators, calculate their prolongations according to Theorem 4.17 and then use Theorem 4.16 to determine the defining equations for the coefficient functions of the infinitesimal generators. The defining equations now yield PDEs in $\mathcal{G}_\tau$. Any solution of these equations that defines a $\mathcal{G}$-$n$-complete generator will upon integration yield a symmetry group in $\mathcal{G}_\tau$.

**4.18 Example** Scalar conservation laws of the form

$$u_t + F(u)u_x = 0 \tag{37}$$

arise in the kinetic theory of traffic flow. Here $u$ denotes the density, and the propagation velocity $F$ may be a strictly decreasing function of $u$ with one or more jumps. A typical case is a unimodal flux function (whose derivative is $F$) with a kink at its maximum, as supported by experimental data [15]. Convolution with a nonnegative mollifier $(\rho_\varepsilon)_{\varepsilon \in I}$ allows to interpret $F$ as an element of $\mathcal{G}_\tau(\mathbb{R})$ which is invertible. Thus our theory of symmetry transformations for equations with generalized nonlinearities applies. The determining equations are

$$\varphi_t + F\varphi_x = 0$$
$$-\xi_x + F\tau_t + \tau F_t + \varphi F_u - F\xi_x + F^2\tau_x + \xi F_x = 0$$

with infinitesimal generator $\mathbf{v} = \xi(x,t)\partial_x + \tau(x,t)\partial_t + \varphi(x,t,u)\partial_u$. As a particular solution we obtain $\mathbf{v} = xt\partial_x + t^2\partial_t + (F'(u))^{-1}(x - tF(u))\partial_u$. The corresponding

generalized group action can be calculated explicitly in $\mathcal{G}_\tau$ showing that if $u$ is a $\mathcal{G}_\tau$ -solution to (37) then so is

$$(x,t) \to F^{-1}\left(\eta x(1+\eta t)^{-1} + F(u(x(1+\eta t)^{-1}, t(1+\eta t)^{-1})(1+\eta t)^{-1}\right)$$

In particular, a constant state $u$ is transformed into a generalized solution to (37) which, depending on the shape of $F$, will generally be associated with a piecewise smooth function.

**4.19 Example** The nonlinear d'Alembert-Hamilton system

$$\begin{aligned} u_{tt} - u_{xx} - u_{yy} - u_{zz} = F(u) \\ u_t^2 - u_x^2 - u_y^2 - u_z^2 = G(u) \end{aligned} \tag{38}$$

arises in the study of relativistic field equations [7] and as a constraint in reducing the nonlinear wave equation to an ODE [12, 13]. One of its symmetries is generated by the vector field $\mathbf{v} = \varphi(u)\partial_u$ where the function $\varphi$ has to satisfy

$$\begin{aligned} F\varphi_u - \varphi F_u + G\varphi_{uu} = 0 \\ 2G\varphi_u - \varphi G_u = 0\,. \end{aligned}$$

In particular, in the isotropic case $F \equiv G \equiv 0$ the function $\varphi$ is arbitrary. In our theory it may be taken in $\mathcal{G}_\tau(\mathbb{R})$ subject to the $\mathcal{G}$-completeness conditions formulated above. As an example of the possible behavior of generalized transformations, consider the vectorfield $\mathbf{v} = \varphi(u)\partial_u$ where $\varphi \in \mathcal{G}_\tau(\mathbb{R})$ is the class of $(\varphi_\varepsilon)_{\varepsilon\in I}$ with $\varphi_\varepsilon(u) = \tanh(\frac{u}{\varepsilon})$. Thus $\varphi(u)$ is associated with the jump function $-\mathrm{sgn}(u)$. Starting with a classical smooth solution $u = u(x,t) \in \mathcal{O}_C(\mathbb{R}^4)$ of the isotropic d'Alembert-Hamilton system ((38) with $F \equiv G \equiv 0$), the generalized symmetry transform generated by the vector field $\mathbf{v}$ turns $u(x,t)$ into the generalized solution $\tilde{U} \in \mathcal{G}_\tau(\mathbb{R}^4)$ with representative

$$\tilde{u}_\varepsilon(x,t) = \varepsilon\,\mathrm{Arsinh}\left(e^{\eta/\varepsilon}\sinh\frac{u(x,t)}{\varepsilon}\right)\,.$$

When $\eta > 0$, it is straightforward to check that $\tilde{U}$ is associated with the piecewise smooth function $v(x,t) = u(x,t) + \eta\,\mathrm{sgn}(u(x,t))$. The generalized symmetry this way transforms smooth solutions into discontinuous solutions.

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